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On Simultaneous Best L_1 Approximations in $C[-1, 1]$

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This paper gives the following result. Let V_1 and V_2 be Chebyshev subspaces of $C[-1, 1]$ with dimensions 1 and n ($n > 1$), respectively. Let $V_1 \subset V_2$ and $v_j \in V_j$ ($j = 1, 2$). Then there exists an $f \in C[-1, 1]$ such that v_j is a best L_1 approximation to f from V_j ($j = 1, 2$) if and only if $v = v_2 - v_1$ changes sign at least once in $[-1, 1]$ or is equal to zero.

1. INTRODUCTION

In a conference held at Oberwolfach in 1968, Rivlin [1] proposed the following problem ($\equiv [-1, 1]$):

Characterize those n-tuples of algebraic polynomials $\{p_0, p_1, ..., p_{n-1}\}$ with degrees satisfying

$$\deg p_j = j \qquad (j = 0, 1, ..., n-1),$$

for which there exists an $f \in C(X)$ such that the polynomial of best uniform approximation of degree j to f is p_j ($j = 0, 1, ..., n-1$). What is the answer in the particular case $n = 2$?

Several authors have studied this problem (see the references in [2]). In [6] the author has considered the same problem in $C(X)$ with the L_1 norm

$$\|f\| = \int_X |f(x)|\, dx$$

and has given the answer in the particular case when $n = 2$: There exists an $f \in C(X)$ such that p_j ($j = 0, 1$) is a best approximation to f if and only if the polynomial $p = p_1 - p_0$ changes sign once. In this paper we generalize this result to the case of two prescribed best approximations $v_j \in V_j$

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$(j=1,2)$, where V_1 and V_2 are Chebyshev subspaces of $C(X)$ with dimensions 1 and n $(n>1)$, respectively.

THEOREM. *Let V_1 and V_2 be Chebyshev subspaces of $C(X)$ with dimensions 1 and n $(n>1)$, respectively. Let $V_1 \subset V_2$ and $v_j \in V_j$ $(j=1,2)$. Then there exists an $f \in C(X)$ such that v_j is a best approximation to f from V_j $(j=1,2)$ if and only if the function $v=v_2-v_1$ changes sign at least once in X or is indentically equal to zero.*

Before proving the theorem we introduce some notation. For $g \in C(X)$ write

$$Z_+(g)=\{x \in X: g(x)>0\},$$
$$Z_-(g)=\{x \in X: g(x)<0\},$$
$$Z(g)=\{x \in X: g(x)=0\}.$$
$m(E)=$ the Lebesgue measure of the set E.

2. PROOF OF THE THEOREM

We can suppose without loss of generality that $v_1=0$.

Necessity. Assume that there exists an $f \in C(X)$ such that v_j is a best approximation to f from V_j $(j=1,2)$, where $v_1=0$, but the condition of the theorem is not satisfied, i.e., v $(=v_2)$ $\neq 0$ and does not change sign in X, say $v \geqslant 0$ on X.

Letting $u \in V_1$ and $u>0$, by Theorem 4-2 in [3] we have

$$\left| \int_X u \operatorname{sgn} f \, dx \right| \leqslant \int_{Z(f)} u \, dx,$$

$$\left| \int_X u \operatorname{sgn}(f-v) \, dx \right| \leqslant \int_{Z(f-v)} u \, dx,$$

i.e.

$$\left| \int_{Z_+(f)} u \, dx - \int_{Z_-(f)} u \, dx \right| \leqslant \int_{Z(f)} u \, dx,$$

$$\left| \int_{Z_+(f-v)} u \, dx - \int_{Z_-(f-v)} u \, dx \right| \leqslant \int_{Z(f-v)} u \, dx.$$

Hence

$$\int_{Z_+(f)} u\,dx - \int_{Z_-(f)} u\,dx - \int_{Z(f)} u\,dx \leqslant 0, \tag{1}$$

$$\int_{Z_+(f-v)} u\,dx - \int_{Z_-(f-v)} u\,dx + \int_{Z(f-v)} u\,dx \geqslant 0. \tag{2}$$

On the other hand, since

$$Z_-(f) \cup Z(f) \subset Z_-(f-v) \cup Z(v)$$

and

$$Z_+(f-v) \cup Z(f-v) \subset Z_+(f) \cup Z(v), \tag{3}$$

$$\int_{Z_-(f)} u\,dx + \int_{Z(f)} u\,dx \leqslant \int_{Z_-(f-v)} u\,dx$$

and

$$\int_{Z_+(f-v)} u\,dx + \int_{Z(f-v)} u\,dx \leqslant \int_{Z_+(f)} u\,dx.$$

Whence

$$\int_{Z_+(f-v)} u\,dx + \int_{Z(f-v)} u\,dx - \int_{Z_-(f-v)} u\,dx$$
$$\leqslant \int_{Z_+(f)} u\,dx - \int_{Z(f)} u\,dx - \int_{Z_-(f)} u\,dx. \tag{4}$$

From (1), (2), and (4) it follows that

$$\int_{Z_+(f-v)} u\,dx + \int_{Z(f-v)} u\,dx - \int_{Z_-(f-v)} u\,dx$$
$$= \int_{Z_+(f)} u\,dx - \int_{Z(f)} u\,dx - \int_{Z_-(f)} u\,dx = 0.$$

So

$$\int_{Z_-(f)} u\,dx + \int_{Z(f)} u\,dx = \int_{Z_+(f)} u\,dx = c = \tfrac{1}{2}\int_X u\,dx,$$

$$\int_{Z_+(f-v)} u\,dx + \int_{Z(f-v)} u\,dx = \int_{Z_-(f-v)} u\,dx = c. \tag{5}$$

Thus

$$\int_{Z_+(f-v)} u\,dx + \int_{Z(f-v)} u\,dx = \int_{Z_+(f)} u\,dx. \tag{6}$$

From (3) and (6) we obtain

$$\int_{Z_+(f)\backslash(Z_+(f-v)\cup Z(f-v))} u\,dx = 0,$$

which implies

$$m(Z_+(f)\backslash(Z_+(f-v)\cup Z(f-v))) = 0.$$

Since the set

$$\begin{aligned} Z_+(f)\backslash(Z_+(f-v)\cup Z(f-v)) &= Z_+(f)\cap Z_-(f-v) \\ &= \{x\in X: 0 < f(x) < v(x)\} \end{aligned}$$

is open, $\{x\in X: 0 < f(x) < v(x)\} = \varnothing$. This gives that

$$\begin{aligned} Z^*(v) &\equiv \overline{Z_-(f-v)} \cap (Z_+(f-v)\cup Z(f-v)) \\ &= \overline{Z_-(f-v)}\backslash Z_-(f-v) \\ &\subset Z(v). \end{aligned}$$

But v has at most $n-1$ zeros in X, because $v\in V_2$, where V_2 is an n-dimensional Chebyshev subspace. Thus there exists a $u^*\in V_2$ such that [5, p. 30]

$$\begin{aligned} u^*(x) &< 0, \qquad x\in(Z_+(f-v)\cup Z(f-v))\backslash Z^*(v), \\ &> 0, \qquad x\in Z_-(f-v), \end{aligned}$$

and $\max_{x\in X}|u^*(x)| \leqslant \min_{x\in X} u(x)$, for which by (5) we have

$$\begin{aligned} &\left|\int_X (u+u^*)\operatorname{sgn}(f-v)\,dx\right| \\ &\quad > -\int_X (u+u^*)\operatorname{sgn}(f-v)\,dx - \int_X |u^*|\,dx \\ &\quad = \int_{Z_-(f-v)} u\,dx - \int_{Z_+(f-v)} u\,dx - \int_X u^*\operatorname{sgn}(f-v)\,dx - \int_X |u^*|\,dx \\ &\quad = \int_{Z(f-v)} u\,dx + \int_{Z(f-v)} u^*\,dx \\ &\quad = \int_{Z(f-v)} |u+u^*|\,dx, \end{aligned}$$

a contradiction.

Sufficiency. Assume now that v changes sign at least once in X, because the theorem is obviously valid for $v = 0$.

First we show the theorem for v having one change of sign. Let v change sign at $x^* \in (-1, 1)$.

By Lemma 2 in [4] there exist points

$$-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = x^* < x_{n+2} < \cdots < x_{2n+1} < x_{2n+2} = 1$$

such that

$$\sum_{i=0}^{n} (-1)^i \int_{x_i}^{x_{i+1}} u\,dx = 0, \qquad \forall u \in V_2, \tag{7}$$

$$\sum_{i=n+1}^{2n+1} (-1)^i \int_{x_i}^{x_{i+1}} u\,dx = 0, \qquad \forall u \in V_2. \tag{8}$$

Write $N = [n/2]$, $N' = [(n-1)/2]$ and denote

$$G_1 = \bigcup_{i=0}^{N} [x_{2i}, x_{2i+1}],$$

$$G_2 = \bigcup_{i=N'+1}^{n} [x_{2i+1}, x_{2i+2}],$$

$$E_1 = \bigcup_{i=0}^{N'} [x_{2i+1}, x_{2i+2}],$$

$$E_2 = \bigcup_{i=N+1}^{n} [x_{2i}, x_{2i+1}],$$

$$G = G_1 \cup G_2,$$

$$E = E_1 \cup E_2,$$

$$H = \left(\bigcup_{\substack{i=1 \\ i \neq n+1}}^{2n+1} (x_i - h, x_i + h) \right) \cap E,$$

where $0 < h < \frac{1}{2}\min_{1 \leqslant i \leqslant 2n}(x_{i+1} - x_i)$. With this notation (7) and (8) become

$$\int_{G_1} u\,dx = \int_{E_1} u\,dx, \qquad \forall u \in V_2 \tag{9}$$

and

$$\int_{G_2} u\,dx = \int_{E_2} u\,dx, \qquad \forall u \in V_2. \tag{10}$$

Now put

$$
\begin{aligned}
f(x) &= v(x), && x \in G,\\
&= 0, && x \in E\backslash H,\\
&= \text{a continuous curve, the points in which almost everywhere strictly lie between 0 and v,} && x \in [x_i - h, x_i + h] \cap E\\
& && (i = 1,..., n, n+2,..., 2n+1).
\end{aligned}
$$

Take $x < x^*$ such that $v(x) \neq 0$ and let $s = \operatorname{sgn} v(x)$. Whence we have almost everywhere

$$
\begin{aligned}
\operatorname{sgn} f(x) &= \quad s, && x \in G_1 \cup (E_1 \cap H),\\
&= -s, && x \in G_2 \cup (E_2 \cap H),\\
&= 0, && x \in E\backslash H,\\
\operatorname{sgn}(f(x) - v(x)) &= -s, && x \in E_1,\\
&= \quad s, && x \in E_2,\\
&= 0, && x \in G.
\end{aligned}
$$

Thus since by (9) and (10) for any $u \in V_2$

$$
\begin{aligned}
\left| \int_X u \operatorname{sgn}(f-v)\, dx \right| &= \left| \int_{E_1} u\, dx - \int_{E_2} u\, dx \right| = \left| \int_{G_1} u\, dx - \int_{G_2} u\, dx \right|\\
&\leqslant \int_{G_1} |u|\, dx + \int_{G_2} |u|\, dx\\
&= \int_G |u|\, dx = \int_{Z(f-v)} |u|\, dx,
\end{aligned}
$$

v is a best approximation to f from V_2.

On the other hand, for any $u \in V_1$

$$
\begin{aligned}
\left| \int_X u \operatorname{sgn} f\, dx \right| &= \left| s\int_{G_1} u\, dx - s\int_{G_2} u\, dx + \int_H u \operatorname{sgn} f\, dx \right|\\
&= \left| s\int_{E_1} u\, dx - s\int_{E_2} u\, dx + \int_H u \operatorname{sgn} f\, dx \right|\\
&= \left| s\int_{E_1\backslash H} u\, dx - s\int_{E_2\backslash H} u\, dx + 2\int_H u \operatorname{sgn} f\, dx \right|\\
&\leqslant \int_{E_1\backslash H} |u|\, dx + \int_{E_2\backslash H} |u|\, dx = \int_{E\backslash H} |u|\, dx = \int_{Z(f)} |u|\, dx
\end{aligned}
$$

provided $h > 0$ is small enough (in fact independent of u of V_1), because u does not change sign and $m(E_j) > 0$ $(j = 1, 2)$. This means 0 is a best approximation to f from V_1.

Therefore f is the needed function and meanwhile satisfies

$$f(-1) = v(-1), \qquad f(1) = v(1).$$

Generally, let v change sign at points x_l^*, $l = 1, 2,..., k$ $(k < n)$, $-1 < x_1^* < x_2^* < \cdots < x_k^* < 1$. Take $k + 1$ points $y_0, y_1,..., y_k$ such that

$$-1 = y_0 < x_1^* < y_1 < x_2^* < y_2 < \cdots < x_k^* < y_k = 1.$$

In each subinterval $[y_{l-1}, y_l]$, $l = 1, 2,..., k$, v changes sign exactly once, so according to the result above there exists an $f_l \in C[y_{l-1}, y_l]$ such that v_j is a best approximation to f from V_j $(j = 1, 2)$ in $[y_{l-1}, y_l]$ and

$$f_l(y_{l-1}) = v(y_{l-1}), \qquad f_l(y_l) = v(y_l), \qquad l = 1, 2,..., k,$$

where $v = v_2 - v_1$. Hence

$$f(x) = f_l(x), \qquad x \in [y_{l-1}, y_l], \qquad l = 1, 2,..., k$$

belongs to $C(X)$ and has v_j as a best approximation from V_j $(j = 1, 2)$. This completes the proof of the theorem.

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